Reliable Integer Ambiguity Resolution

Soft Constraints on the Baseline Length and Direction, and New Multi-frequency Code Carrier Linear Combinations

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Abstract

In this paper, a maximum a posteriori probability estimation of baselines and ambiguities is proposed for RTK and attitude determination. The estimator uses statistical a priori information about the baseline length, pitch and heading, and thereby improves the accuracy of the float solution. It is more robust than traditional attitude determination techniques with deterministic baseline constraints. The inertia of the receivers are considered in a movement model, which is integrated into an extended Kalman filter. Moreover, a new set of multi-frequency code carrier linear combinations is derived, which enables an arbitrary scaling of the geometry, an arbitrary scaling of the ionospheric delay, and any preferred wavelength.

Keywords

Attitude determination • Ambiguity resolution • Maximum a posteriori probability estimation • Linear combinations

1 Introduction

Reliable carrier phase integer ambiguity resolution for RTK positioning and attitude determination is today mainly limited by the code multipath which can substantially exceed the carrier wavelength of 19 cm. Low cost single frequency L1 systems will be used in automobile applications, and require some baseline a priori information for reliable fixing. For ship applications, dual frequency systems are integrated with linear combinations that increase the wavelength but also amplify the multipath from nearby containers (Fig. 1). Therefore, there is a need for improved linear combinations.

Teunissen's Least-squares Ambiguity Decorrelation Adjustment (LAMBDA) method (Teunissen 1995) is widely used for ambiguity resolution: it includes an integer decorrelation and a subsequent search, and provides the integer least-squares solution. Teunissen also developed a constrained LAMBDA algorithm for attitude determination (Teunissen 2010), which uses either deterministic or stochastic a priori information about the baseline *length*.

The purpose of this paper is fourfold: first, a maximum a posteriori (MAP) probability estimation of baselines and ambiguities is suggested. It uses statistical a priori information about the baseline *length*, *heading* and *pitch*, which substantially reduces the search space volume as shown in Fig. 2. Secondly, a group of multi-frequency linear combinations is derived, which include both code and carrier phase measurements and enable an arbitrary scaling of geometry and ionospheric delay, as well as any preferred wavelength. The third objective is the inclusion of a movement model into an extended Kalman filter, and the fourth objective is an inequality constrained estimation of the float solution with the barrier method.

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Fig. 1 Attitude determination with GNSS: the heading can be determined with at least two GNSS receivers. The length and pitch angle of the baseline are tightly constrained, which enables a reliable resolution of carrier phase ambiguities

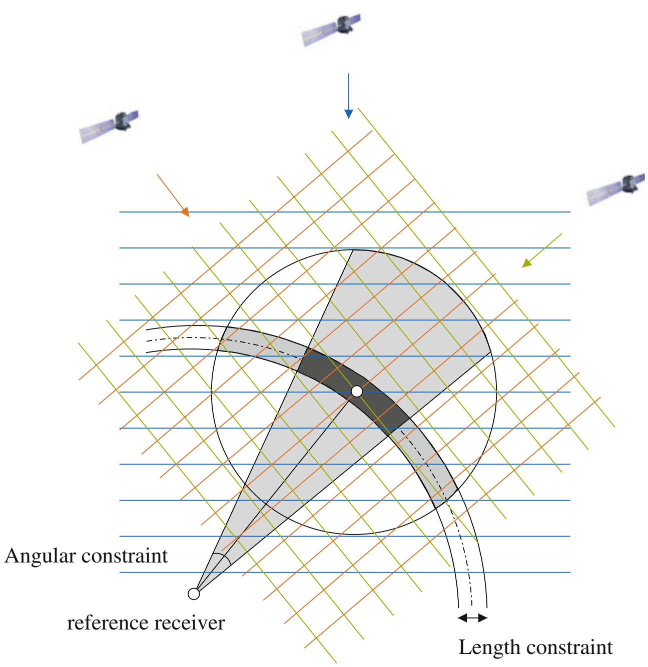

Fig. 2 Integer ambiguity resolution with attitude a priori information: the periodic wavefronts from three satellites intersect in an infinite number of points. The unambiguous code measurements and a priori information about the baseline length and pitch enable a substantial reduction of the search space

2 Traditional Attitude Determination with GNSS

The double difference code and carrier phase measurements of $K+1$ satellites and a baselines can be written in a matrix of $2K$ rows and a columns, and were modeled by Teunissen (2010) as

$$z = \boldsymbol{H}\boldsymbol{R}\boldsymbol{\xi}' + \boldsymbol{A}\boldsymbol{N} + \boldsymbol{\varepsilon} \quad \text{s.t.} \quad \boldsymbol{R} \in \mathbb{O}^{3\times p}, \boldsymbol{N} \in \mathbb{Z}^{K\times a}, \tag{1}$$

with the geometry matrix $\boldsymbol{H}$, the known baselines $\boldsymbol{\xi}'$ in the local body-fixed coordinate frame, the unknown orthonormal rotation matrix $\boldsymbol{R}$ for rotating the local coordinate frame into the ECEF by p angular rotations, the ambiguity prefactor matrix $\boldsymbol{A}$, the integer ambiguities $\boldsymbol{N}$ and the combined noise/multipath term $\boldsymbol{\varepsilon}$. The matrix equation (1) can be brought into a vector equation using the property $\text{vec}(\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{C}^T \otimes \boldsymbol{A})\text{vec}(\boldsymbol{B})$:

$$\begin{aligned} \text{vec}(z) = & \underbrace{[\mathbf{1}^{a\times a} \otimes \boldsymbol{A}]}_{\boldsymbol{A}'} \text{vec}(\boldsymbol{N}) + \underbrace{[(\boldsymbol{\xi}')^T \otimes \boldsymbol{H}]}_{\boldsymbol{H}'} \text{vec}(\boldsymbol{R}) \\ & + \text{vec}(\boldsymbol{\varepsilon}), \end{aligned} \tag{2}$$

with $\mathbf{1}^{a\times a}$ being the unit matrix with a columns and rows. If the orthonormality of $\boldsymbol{R}$ is ignored, its estimation can be separated from the ambiguity estimation by an orthogonal projection $\boldsymbol{P}^{\perp}_{\boldsymbol{A}'}$ on the space of $\boldsymbol{A}'$, i.e.

$$\boldsymbol{P}^{\perp}_{\boldsymbol{A}'}\text{vec}(z) = \boldsymbol{P}^{\perp}_{\boldsymbol{A}'}\boldsymbol{H}'\text{vec}(\boldsymbol{R}) + \boldsymbol{P}^{\perp}_{\boldsymbol{A}'}\text{vec}(\boldsymbol{\varepsilon}). \tag{3}$$

The least-squares estimate of the re-arranged rotation matrix $\text{vec}(\boldsymbol{R})$ then follows as

$$\text{vec}(\hat{\boldsymbol{R}}) = \left(\bar{\boldsymbol{H}}^T \boldsymbol{\Sigma}^{-1} \bar{\boldsymbol{H}}\right)^{-1} \bar{\boldsymbol{H}}^T \boldsymbol{\Sigma}^{-1} \text{vec}(z), \tag{4}$$

with the measurement noise covariance matrix $\boldsymbol{\Sigma}$ and

$$\bar{\boldsymbol{H}} = \boldsymbol{P}^{\perp}_{\boldsymbol{A}'}\left[(\boldsymbol{\xi}')^T \otimes \boldsymbol{H}\right]. \tag{5}$$

An estimate of the orthonormality constrained rotation matrix is obtained from the unconstrained one of (2) by

$$\check{\boldsymbol{R}} = \text{argmin}_{\boldsymbol{R}\in\mathbb{O}^{3\times p}} \|\text{vec}(\hat{\boldsymbol{R}} - \boldsymbol{R})\|^2_{\boldsymbol{Q}^{-1}_{\hat{\boldsymbol{R}}}}, \tag{6}$$

which can also be written as a Lagrange optimization

$$\begin{aligned} \check{\boldsymbol{R}} = & \arg\min_{\boldsymbol{R}} \left(\|\text{vec}(\hat{\boldsymbol{R}} - \boldsymbol{R})\|^2_{\boldsymbol{Q}^{-1}_{\hat{\boldsymbol{R}}}} \right. \\ & \left. + \text{tr}\left(\boldsymbol{\Lambda}\left(\boldsymbol{R}^T\boldsymbol{R} - \mathbf{1}^{p\times p}\right)\right)\right), \end{aligned} \tag{7}$$

where the Lagrangian multiplier $\boldsymbol{\Lambda}$ is given by Teunissen for $p = 3$ in Teunissen (2010) by

$$\Lambda = \begin{bmatrix} \lambda_1 & \frac{1}{2}\lambda_4 & \frac{1}{2}\lambda_5 \\ \frac{1}{2}\lambda_4 & \lambda_2 & \frac{1}{2}\lambda_6 \\ \frac{1}{2}\lambda_5 & \frac{1}{2}\lambda_6 & \lambda_3 \end{bmatrix}. \tag{8}$$

The reliability of ambiguity resolution can be improved if an a priori knowledge about the rotation matrix is available. In this case, the measurement vector is extended by the a priori information $\bar{\boldsymbol{R}}$ on $\boldsymbol{R}$, i.e.

$$\begin{bmatrix} \mathrm{vec}(z) \\ \mathrm{vec}(\bar{\boldsymbol{R}}) \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{I}^{a\times a} \otimes \boldsymbol{A} \\ \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{A}'} \mathrm{vec}(\boldsymbol{N}) + \underbrace{\begin{bmatrix} (\boldsymbol{\xi}')^T \otimes \boldsymbol{H} \\ \boldsymbol{1}^{3\times 3} \end{bmatrix}}_{\boldsymbol{H}'} \mathrm{vec}(\boldsymbol{R}) + \mathrm{vec}(\boldsymbol{\varepsilon}). \quad (9)$$

In practice, the baseline/ attitude a priori information is given in spherical parameters (length, pitch and heading) rather than in the rotation matrix itself. The rotation matrix $\bar{\boldsymbol{R}}$ is then obtained from the relation

$$\bar{\boldsymbol{R}}\boldsymbol{\xi}' = \boldsymbol{R}_{\substack{\mathrm{ENU}\\ \mathrm{ECEF}}} \begin{bmatrix} \bar{l}\cos(\bar{\nu}_1)\cos(\bar{\nu}_2) \\ \bar{l}\cos(\bar{\nu}_1)\cos(\bar{\nu}_2) \\ \bar{l}\sin(\bar{\nu}_1) \end{bmatrix}, \quad (10)$$

where $\boldsymbol{R}_{\substack{\mathrm{ENU}\\ \mathrm{ECEF}}}$ describes the known rotation from ENU into ECEF coordinates, and $\bar{l}$, $\bar{\nu}_1$ and $\bar{\nu}_2$ denotes the a priori information about the baseline length, elevation and heading. This transformation of spherical baseline parameters into a rotation matrix has one major disadvantage: it is not feasible to determine efficiently the statistics of $\bar{\boldsymbol{R}}$ from the statistics of $\bar{l}$, $\bar{\nu}_1$ and $\bar{\nu}_2$ due to the trigonometric functions.

3 MAP Estimation of Attitude and Integer Ambiguities

In maritime applications, the attitude and the rotation of the body w.r.t. the local ENU-frame are of great interest. Therefore, a state vector shall be introduced, which includes the baseline length, the direction angles and their drifts at epoch n as well as the integer ambiguities, i.e.

$$\boldsymbol{x}_n = \left[l_n, \nu_{1,n}, \dot{\nu}_{1,n}, \nu_{2,n}, \dot{\nu}_{2,n}, \boldsymbol{N}^T \right]^T. \quad (11)$$

A Maximum Likelihood (ML) estimator tries to find the baseline parameters l_n, $\nu_{1,n}$, $\dot{\nu}_{1,n}$, $\nu_{2,n}$, $\dot{\nu}_{2,n}$ and integer ambiguities $\boldsymbol{N}$, which have generated the measurements z_n with highest likelihood, i.e.

$$\max_{\boldsymbol{x}_n} p(z_n|\boldsymbol{x}_n). \quad (12)$$

An even more powerful estimator is the maximum a posteriori (MAP) probability estimator, which maximizes the probability of $\nu_{1,n}$, $\dot{\nu}_{1,n}$, $\nu_{2,n}$, $\dot{\nu}_{2,n}$ and l_n for a given z_n. The MAP estimator is defined as

$$\max_{\boldsymbol{x}_n} p(\boldsymbol{x}_n|z_n) = \max_{\boldsymbol{x}_n} p(z_n|\boldsymbol{x}_n) \cdot \frac{p(\boldsymbol{x}_n)}{p(z_n)}, \quad (13)$$

where the rule of Bayes was applied. The conditional probability density on the right side describes the measurement noise, which is typically assumed to be Gaussian distributed:

$$p(z_n|\boldsymbol{x}_n) = \frac{1}{\sqrt{(2\pi)^{2K}|\boldsymbol{\Sigma}_n|}} e^{-\frac{1}{2}\|z_n - \boldsymbol{h}_n(\boldsymbol{x}_n)\|^2_{\boldsymbol{\Sigma}_n^{-1}}}, \quad (14)$$

where $\boldsymbol{\Sigma}_n$ describes the covariance matrix of the measurement noise. The probability density in the nominator of (13) describes the statistical a priori information about the baseline length/ orientation and integer ambiguities. As the ambiguities are integer-valued and as there is in general no statistical a priori information about ambiguities available, we perform a search of integer candidates and a subsequent constrained MAP estimation for each integer candidate. The attitude a priori information shall be modeled by statistically independent Gaussian distributions, i.e.

$$p(\tilde{\boldsymbol{x}}_n) = p(l_n)p(\nu_{1,n})p(\dot{\nu}_{1,n})p(\nu_{2,n})p(\dot{\nu}_{2,n})p(l_n), \quad (15)$$

with $\tilde{\boldsymbol{x}}_n$ being $\boldsymbol{x}_n$ excluding ambiguities, and

$$p(l_n) = \frac{1}{\sqrt{2\pi\sigma^2_{l_n}}} e^{-\frac{(l_n - \bar{l}_n)^2}{2\sigma^2_{l_n}}},$$

$$p(\nu_{i,n}) = \frac{1}{\sqrt{2\pi\sigma^2_{\nu_{i,n}}}} e^{-\frac{(\nu_{i,n} - \bar{\nu}_{i,n})^2}{2\sigma^2_{\nu_{i,n}}}}, \quad i \in \{1,2\}$$

$$p(\dot{\nu}_{i,n}) = \frac{1}{\sqrt{2\pi\sigma^2_{\dot{\nu}_{i,n}}}} e^{-\frac{(\dot{\nu}_{i,n} - \bar{\dot{\nu}}_{i,n})^2}{2\sigma^2_{\dot{\nu}_{i,n}}}}, \quad i \in \{1,2\}.$$

The probability density of $z_n - \boldsymbol{A}\boldsymbol{N}^c$ for integer candidate $\boldsymbol{N}^c$ can be considered as a marginal distribution:

$$p(z_n - \boldsymbol{A}\boldsymbol{N}^c) = \int\int\int\int\int p\left(z_n - \boldsymbol{A}\boldsymbol{N}^c|\tilde{\boldsymbol{x}}_n\right) p(l_n)p(\nu_{1,n})p(\dot{\nu}_{1,n})p(\nu_{2,n})p(\dot{\nu}_{2,n}) \, dl_n d\nu_{1,n} d\dot{\nu}_{1,n} d\nu_{2,n} d\dot{\nu}_{2,n}. \quad (16)$$

3.1 Multi-frequency Code Carrier Linear Combinations

The measurements on multiple frequencies can be linear combined, such that a combination ambiguity with significantly increased wavelength is generated and resolved by the MAP estimator much more reliably than the individual

Table 1 Multi-frequency code carrier widelane Galileo combinations of maximum discrimination for $\sigma_{\varphi^k_{u,m}} = 1$ mm and $\sigma_{\rho^k_{u,m}}$ being equal to the Cramer Rao Bound for $C/N_0 = 45$ dB-Hz

h_1	h_2	E1		E6		E5b		E5a		λ	σ
1	0	j_1	1			j_2	−4	j_3	3		
		α_1	18.9326			α_2	−58.0271	α_3	42.4139	3.603 m	13.9 cm
		β_1	−0.2871			β_2	−0.9899	β_3	−1.0423		
1	0	j_1	1	j_2	−3	j_3	0	j_4	2		
		α_1	21.0108	α_2	−51.1627	α_3	0	α_4	31.3798	3.998 m	6.5 cm
		β_1	−0.0239	β_2	−0.0349	β_3	−0.0824	β_4	−0.0867		
1	−0.1	j_1	1	j_2	−3	j_3	0	j_4	2		
		α_1	19.7197	α_2	−48.0187	α_3	0	α_4	29.4514	3.753 m	6.0 cm
		β_1	−0.0154	β_2	−0.0233	β_3	−0.0554	β_4	−0.0585		
0	−1	j_1	−1	j_2	4	j_3	−1	j_4	−2		
		α_1	−13.1658	α_2	42.7460	α_3	−10.0881	α_4	−19.6632	2.505 m	5.1 cm
		β_1	0.0285	β_2	0.0274	β_3	0.0576	β_4	0.0576		

phase ambiguities. The combination shall include the phase and code measurements of satellite k observed by user u on M frequencies at epoch n, i.e.

$$\lambda \varphi^k_{u,n} = \sum_{m=1}^{M} \left(\alpha_m \lambda_m \varphi^k_{u,m,n} + \beta_m \rho^k_{u,m,n} \right), \tag{17}$$

where the phase coefficients α_m and code coefficients β_m are chosen in Henkel (2012) such that the ambiguity discrimination is maximized, i.e.

$$\max_{\alpha_1,\ldots,\alpha_M,\beta_1,\ldots,\beta_M} \lambda/(2\sigma^k_u), \tag{18}$$

with $(\sigma^k_u)^2 = \sum_{m=1}^{M} \left(\alpha_m^2 \sigma^2_{\varphi^k_{u,m}} + \beta_m^2 \sigma^2_{\rho^k_{u,m}} \right)$. This maximization is in general constrained by a few requirements: the non-dispersive geometry terms shall be scaled by a predefined value h_1 (Henkel and Günther 2012):

$$\sum_{m=1}^{M} (\alpha_m + \beta_m) \stackrel{!}{=} h_1, \tag{19}$$

and the ionospheric delay by any predefined h_2 (Henkel and Günther 2012):

$$\sum_{m=1}^{M} (\alpha_m - \beta_m)\, q^2_{1m} \stackrel{!}{=} h_2, \tag{20}$$

where q_{1m} is the ratio between frequencies f_1 and f_m. Moreover, the ambiguity combination shall correspond to a single virtual integer ambiguity:

$$\sum_{m=1}^{M} \alpha_m \lambda_m N^k_{u,m} \stackrel{!}{=} \lambda N^k_u, \tag{21}$$

which can be rearranged as

$$\sum_{m=1}^{M} \underbrace{\frac{\alpha_m \lambda_m}{\lambda}}_{= j_m \stackrel{!}{\in} \mathbb{Z}} N^k_{u,m} = N^k_u. \tag{22}$$

Table 1 shows the optimized multi-frequency code carrier linear combinations for Galileo, which are characterized by a wavelength of a few meters and a noise level of several centimeters. The respective dual frequency Galileo and GPS combinations can be found in Henkel and Günther (2012).

3.2 Gaussian Attitude A Priori Information

The MAP estimation of (13) was introduced by the authors in Henkel and Günther (2012) and Jurkowski et al. (2011), and includes some a priori information about the baseline length and attitude. This motivates an extension of the vector of double difference measurements to

$$\underbrace{\begin{bmatrix} \lambda \Delta\varphi^1_n \\ \vdots \\ \lambda\Delta\varphi^K_n \\ \hline \Delta\rho^1_n \\ \vdots \\ \Delta\rho^K_n \\ \hline \bar{l}_n \\ \bar{\nu}_{1,n} \\ \bar{\nu}_{2,n} \end{bmatrix}}_{z_n} = \boldsymbol{h}_n(\boldsymbol{x}_n) + \underbrace{\begin{bmatrix} \varepsilon_{\Delta\varphi^1_n} \\ \vdots \\ \varepsilon_{\Delta\varphi^K_n} \\ \hline \varepsilon_{\Delta\rho^1_n} \\ \vdots \\ \varepsilon_{\Delta\rho^K_n} \\ \hline \varepsilon_{\bar{l}_n} \\ \varepsilon_{\bar{\nu}_{1,n}} \\ \varepsilon_{\bar{\nu}_{2,n}} \end{bmatrix}}_{\boldsymbol{v}_n}, \tag{23}$$

with the double difference operator $\Delta o^k_n = (o^k_{1,n} - o^k_{2,n}) - (o^{K+1}_{1,n} - o^{K+1}_{2,n})$, $o \in \{\rho, \lambda\varphi\}$, and

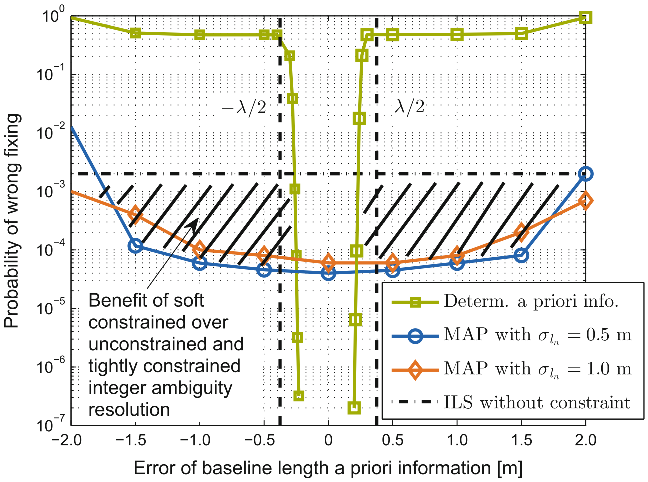

Fig. 3 Benefit of MAP estimation of linear combined ambiguities: the soft constraints enable a lower error rate than both the unconstrained and the deterministically constrained fixing

$$\boldsymbol{h}_n(\boldsymbol{x}_n) = \tag{24}$$

$$\left[\begin{array}{c|ccc|c} (\Delta\boldsymbol{e}_n^1)^T & \lambda & & & \\ \vdots & & \ddots & & \\ (\Delta\boldsymbol{e}_n^K)^T & & & \lambda & \\ \hline (\Delta\boldsymbol{e}_n^1)^T & & & & \\ \vdots & & & & \\ (\Delta\boldsymbol{e}_n^K)^T & & & & \\ \hline & & & & \mathbf{1}^{3\times 3} \end{array}\right] \cdot \begin{bmatrix} l_n\cos(\nu_{1,n})\cos(\nu_{2,n}) \\ l_n\cos(\nu_{1,n})\sin(\nu_{2,n}) \\ l_n\sin(\nu_{1,n}) \\ \boldsymbol{N} \\ l_n \\ \nu_{1,n} \\ \nu_{2,n} \end{bmatrix},$$

where $\Delta\boldsymbol{e}_n^k$ denotes the difference in unit vectors pointing from satellite $k \in \{1,\ldots,K\}$ and reference satellite $K+1$ to the receiver, and is coordinatized in the local East-North-Up (ENU) coordinate frame. The non-linear function $\boldsymbol{h}_n(\boldsymbol{x}_n)$ can be linearized around an initial value $\boldsymbol{x}_0$, which enables an iterative least-squares estimation of the baseline length and attitude.

Figure 3 shows the benefit of MAP ambiguity resolution over both unconstrained and constrained integer least-squares (ILS) estimation with deterministic a priori information. The probability of wrong fixing was determined by Monte-Carlo simulation of an exhaustive search for a dual-frequency E1-E5 phase-only widelane combination with $\lambda = 75.1$ cm, eight visible satellites and four subsequent epochs. Phase multipath was simulated as a Gaussian distributed bias with a variance of $(1\text{ cm})^2$. Obviously, the stochastic a priori knowledge dramatically improves the robustness of the ambiguity resolution. The MAP estimation outperforms ILS both without and with deterministic a priori information.

3.3 State Space Model for Attitude and Ambiguities

The MAP estimation is considered to be especially beneficial for maritime and aerospace applications, where torsional forces modify the baselines. The reliability of ambiguity resolution can be further improved by exploiting the inertia of the vehicle. Therefore, a state space model is introduced:

$$\underbrace{\begin{bmatrix} l_n \\ \nu_{1,n} \\ \dot{\nu}_{1,n} \\ \nu_{2,n} \\ \dot{\nu}_{2,n} \\ \boldsymbol{N} \end{bmatrix}}_{\boldsymbol{x}_n} = \boldsymbol{\Phi}_n \underbrace{\begin{bmatrix} l_{n-1} \\ \nu_{1,n-1} \\ \dot{\nu}_{1,n-1} \\ \nu_{2,n-1} \\ \dot{\nu}_{2,n-1} \\ \boldsymbol{N} \end{bmatrix}}_{\boldsymbol{x}_{n-1}} + \underbrace{\begin{bmatrix} w_{l_n} \\ w_{\nu_{1,n}} \\ w_{\dot{\nu}_{1,n}} \\ w_{\nu_{2,n}} \\ w_{\dot{\nu}_{2,n}} \\ w_{\boldsymbol{N}} \end{bmatrix}}_{\boldsymbol{w}_n}, \quad \boldsymbol{N} \in \mathbb{Z}^{K\times 1}, \tag{25}$$

with the state transition matrix

$$\boldsymbol{\Phi}_n = \left[\begin{array}{c|cc|cc|c} 1 & & & & & \\ \hline & 1 & \delta t & & & \\ & & 1 & & & \\ \hline & & & 1 & \delta t & \\ & & & & 1 & \\ \hline & & & & & \mathbf{1}^{3\times 3} \end{array}\right], \tag{26}$$

with the time interval δt between 2 successive epochs.

3.4 MAP Attitude Determination with Extended Kalman Filter

The state space model and the non-linear measurement model shall now be used in an extended Kalman filter for MAP estimation of baselines and ambiguities.

The extended Kalman filter performs alternating state predictions and updates. The predicted state estimate at epoch n is obtained from the a posteriori estimate $\hat{\boldsymbol{x}}_{n-1}^+$ of the previous epoch using (25), i.e.

$$\hat{\boldsymbol{x}}_n^- = \boldsymbol{\Phi}_n\hat{\boldsymbol{x}}_{n-1}^+, \quad \boldsymbol{P}_{\hat{\boldsymbol{x}}_n^-} = \boldsymbol{\Phi}_n\boldsymbol{P}_{\hat{\boldsymbol{x}}_{n-1}^+}\boldsymbol{\Phi}_n^T + \boldsymbol{Q}_n, \tag{27}$$

with $\boldsymbol{P}_{\hat{\boldsymbol{x}}_n^-}$ being the covariance matrix of $\hat{\boldsymbol{x}}_n^-$ and $\boldsymbol{Q}_n$ being the process noise covariance matrix. Once the measurement of epoch n is available, the non-linear residual $z_n - \boldsymbol{h}_n(\hat{\boldsymbol{x}}_n^-)$ can be computed and the state estimate is updated. The a posteriori state estimate is given by

$$\hat{\boldsymbol{x}}_n^+ = \hat{\boldsymbol{x}}_n^- + \boldsymbol{K}_n\left(z_n - \boldsymbol{h}_n(\hat{\boldsymbol{x}}_n^-)\right), \tag{28}$$

where the Kalman gain $\boldsymbol{K}_n$ is chosen such that the variance of the a posteriori state estimate is minimized:

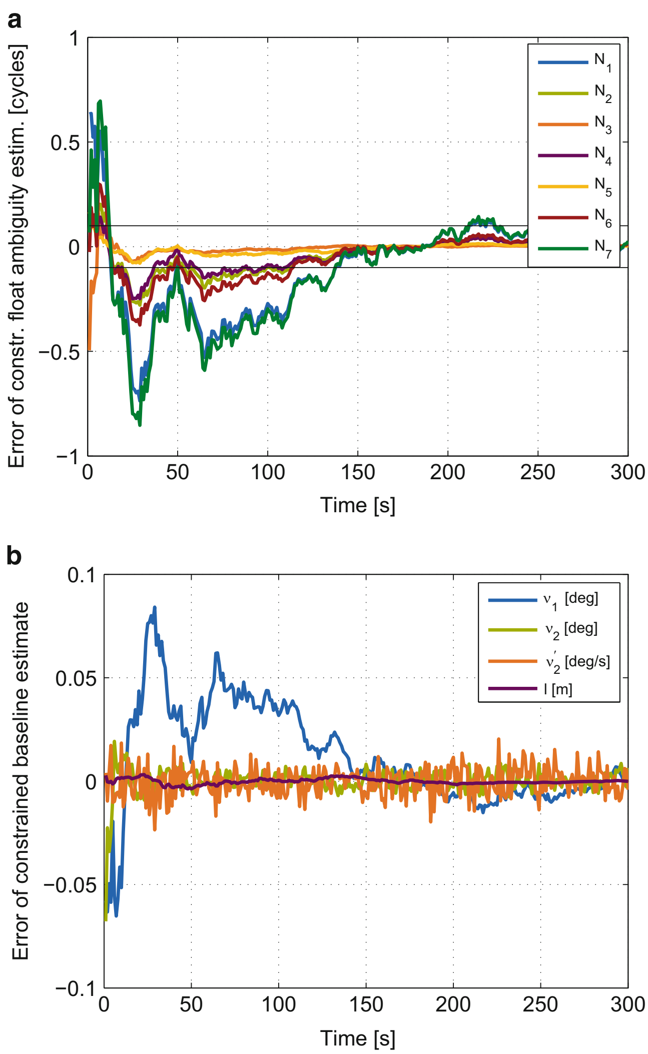

Fig. 4 MAP estimation using Gaussian a priori information with $\sigma_{l_n} = 5\,\text{cm}$, and $\sigma_{\nu_{1,n}} = \infty$. (**a**) Convergence of float ambiguity estimates. (**b**) Convergence of baseline estimates

$$\boldsymbol{K}_n = \arg\min_{\boldsymbol{K}_n} \mathrm{E}\{\|\hat{\boldsymbol{x}}_n^+ - \mathrm{E}\{\hat{\boldsymbol{x}}_n^+\}\|^2\}. \tag{29}$$

As this minimization involves the statistics of a non-linear function $\boldsymbol{h}_n(\boldsymbol{x}_n)$ of random variables, one performs a linearization around an initial estimate $\boldsymbol{x}_0$:

$$\begin{aligned} \boldsymbol{h}_n(\boldsymbol{x}_n) = \boldsymbol{h}_n(\boldsymbol{x}_0) &+ \left.\frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \nu_1}\right|_{\boldsymbol{x}=\boldsymbol{x}_0} (\nu_{1,n} - \nu_{1,0}) \\ &+ \left.\frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \nu_2}\right|_{\boldsymbol{x}=\boldsymbol{x}_0} (\nu_{2,n} - \nu_{2,0}) \\ &+ \left.\frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial l}\right|_{\boldsymbol{x}=\boldsymbol{x}_0} (l_n - l_0) + \ldots \quad . \end{aligned} \tag{30}$$

Figure 4 shows the performance of the MAP estimation for single frequency measurements from eight satellites, a phase noise standard deviation of 1 cm, a code noise standard deviation of 1 m, a baseline length of 100 m and Gaussian a priori knowledge only for the baseline length. One can observe that the heading and pitch angle can be determined with an accuracy of 0.01°.

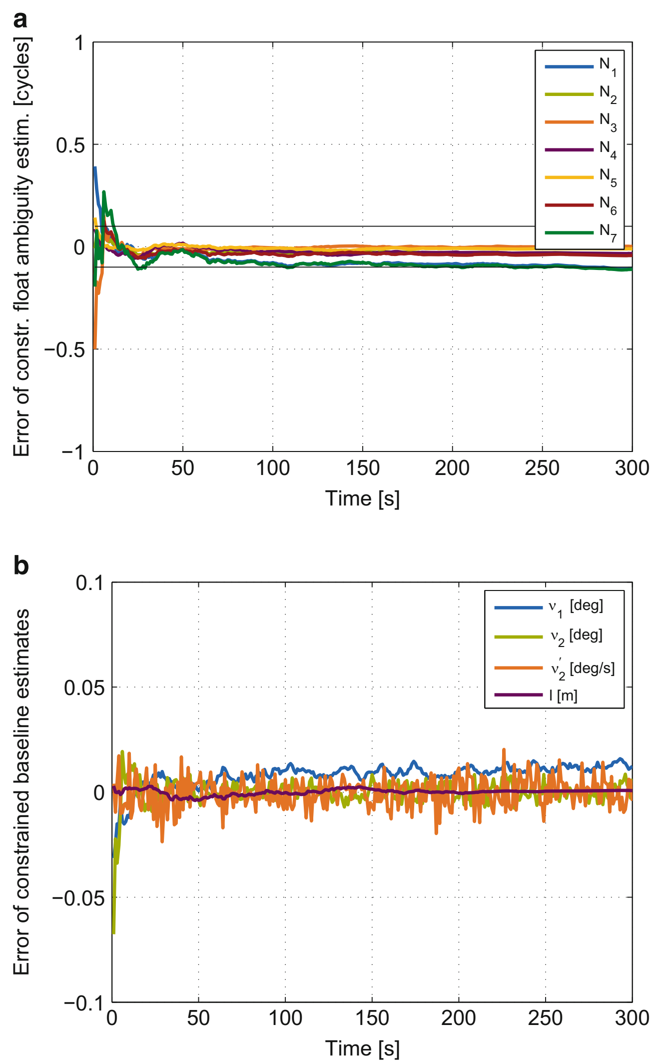

Fig. 5 MAP estimation using Gaussian a priori information with $\sigma_{l_n} = 5\,\text{cm}$, and $\sigma_{\nu_{1,n}} = 0.1°$. (**a**) Convergence of float ambiguity estimates. (**b**) Convergence of baseline estimates

The convergence behaviour shown in Fig. 5 refers to the same setting except for an additional Gaussian a priori information about the pitch angle, which significantly shortens the convergence time and enables a reliable fixing within a few epochs despite the large noise levels.

3.5 Uniformly Distributed Attitude A Priori Information

In some applications, Gaussian a priori information is not available. However, lower and upper bounds can still be

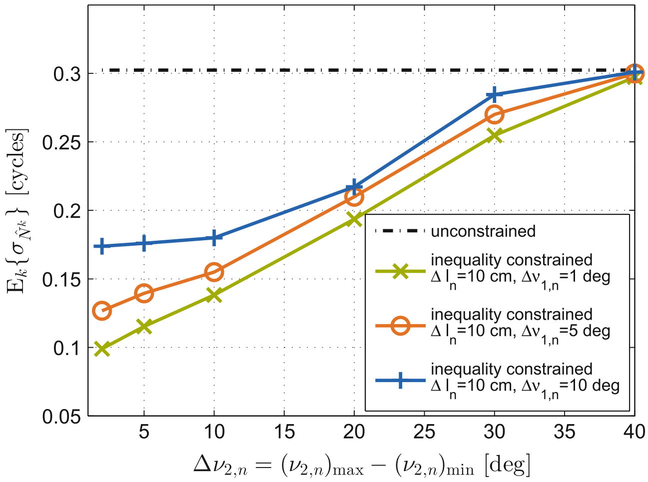

Fig. 6 Inequality constrained widelane ambiguity estimation: both length and angular constraints enable a substantial improvement of the float solution

formulated for the baseline length and orientation, which results in the optimization problem

$$\min_{\nu_{1,n},\nu_{2,n},l_n,\boldsymbol{N}} \|\boldsymbol{z}_n - \boldsymbol{h}_n(\boldsymbol{x}_n) - \boldsymbol{A}\boldsymbol{N}\|^2_{\boldsymbol{\Sigma}_n^{-1}}$$

$$\text{s.t.} \quad (\nu_{i,n})_{\min} \leq \nu_{i,n} \leq (\nu_{i,n})_{\max}, \quad i \in \{1,2\},$$
$$(l_n)_{\min} \leq l_n \leq (l_n)_{\max}. \tag{31}$$

This inequality constrained problem can be transformed into an unconstrained problem:

$$\min_{\nu_{1,n},\nu_{2,n},l_n,\boldsymbol{N}\in\mathbb{Z}^{K\times 1}} \Bigg(\|\boldsymbol{z}_n - \boldsymbol{h}_n(\boldsymbol{x}_n) - \boldsymbol{A}\boldsymbol{N}\|^2_{\boldsymbol{\Sigma}_n^{-1}} \tag{32}$$
$$+ \sum_{i=1}^{2} f\left(\nu_{i,n}, (\nu_{i,n})_{\{\min,\max\}}\right) + f\left(l_n, (l_n)_{\{\min,\max\}}\right) \Bigg),$$

with the quadratic barrier function

$$f(s, s_{\{\min,\max\}}) = \begin{cases} f \cdot (s_{\min} - s)^2 & s < s_{\min}, \\ 0 & s_{\min} \leq s \leq s_{\max} \\ f \cdot (s - s_{\max})^2 & s > s_{\max}, \end{cases}$$

with penalty factor f. This barrier function is second order continuously differentiable and converges to the ideal barrier for $f \to \infty$. Thus, the minimization of (32) can be again solved iteratively with the Newton method. Figure 6 shows a substantial benefit of the inequality constraints for a baseline with $l_n = 5$ m, $\nu_{1,n} = 0°$, $\nu_{2,n} = 45°$, single epoch measurements from 11 satellites, and 2 IF E1-E5 combinations (Henkel and Günther 2012): a code carrier combination with $\lambda = 3.285$ m and a code-only one.

4 Conclusion

In this paper, a maximum a posteriori probability (MAP) estimation of baselines and integer ambiguities was proposed for RTK and PPP with attitude determination. This MAP estimator uses statistical a priori information about the baseline length and attitude, which substantially improves the conditioning of the problem. Moreover, multi-frequency code carrier linear combinations with wavelengths of several meters were derived to further improve the reliability of ambiguity resolution.

References

Henkel P (2012) Bootstrapping with multi-frequency mixed code carrier linear combinations and partial integer decorrelation in the presence of biases. In: Geodesy for planet earth, Proceedings of International Association of Geodesy Symposia 136, Buenos Aires, Argentina, pp 925–933

Henkel P, Günther C (2012) Reliable integer ambiguity resolution: multi-frequency code carrier linear combinations and statistical a priori knowledge of attitude. Navigation 59(1):61–75

Jurkowski P, Henkel P, Gao G, Günther C (2011) Integer ambiguity resolution with tight and soft baseline constraints for freight stabilization at helicopters and cranes. In: Proceedings of ION ITM, San Diego, USA, pp 336–346

Teunissen P (1995) The least-squares ambiguity decorrelation adjustment: a method for fast GPS ambiguity estimation. J Geodes 70:65–82

Teunissen P (2010) Integer least-squares theory for the GNSS compass. J Geodes 84:433–447