Sequential Best Integer-Equivariant Estimation for Geodetic Network Solutions

A. Brack*, P. Henkel*, and C. Günther*,**
*Technische Universität München (TUM), Munich, Germany
**German Aerospace Center (DLR), Oberpfaffenhofen, Germany

BIOGRAPHIES

Andreas Brack studied electrical engineering and information technology at the Technische Universität München (TUM), Munich, Germany. He received his B.Sc. and M.Sc. degree, both with honors, in 2011 and 2012, respectively. Between 2009 and 2011, he was a member of the faculty program for exceptionally talented students. In October 2012 he joined the Institute for Communications and Navigation at the Technische Universität München, where he is currently working towards his PhD in the field of precise point positioning.

Patrick Henkel received the Bachelor, Master and PhD degree from the Technische Universität München, Munich, Germany. He is now working towards his habilitation in the field of precise point positioning. He was a guest researcher at TU Delft in 2007, and at the GPS Lab at Stanford University, Stanford, in 2008 and 2010. Patrick received the Pierre Contensou Gold Medal at the Intern. Astronautical Congress in 2007, the Bavarian regional prize at the European Satellite Navigation Competition in 2010, and the "Vodafone Förderpreis" for his dissertation in 2011. He is also one of the founders of Advanced Navigation Solutions - ANAVS GmbH.

Christoph Günther studied theoretical physics at the Swiss Federal Institute of Technology in Zurich. He received his diploma in 1979 and completed his PhD in 1984. He worked on communication and information theory at Brown Boveri and Ascom Tech. From 1995, he led the development of mobile phones for GSM and later dual mode GSM/Satellite phones at Ascom. In 1999, he became head of the research department of Ericsson in Nuremberg. Since 2003, he is the director of the Institute of Communication and Navigation at the German Aerospace Center (DLR) and since December 2004, he additionally holds a Chair at the Technische Universität München (TUM). His research interests are in satellite navigation, communication and signal processing.

ABSTRACT

The key to high precision parameter estimation (e.g., positioning) in global navigation satellite system (GNSS) applications is to take the integer nature of the carrier-phase ambiguities into account. The class of integer estimators, like integer bootstrapping (BS) or integer least-squares (ILS), fixes the ambiguities to integer values, which can also decrease the precision of the estimates of the non-ambiguity parameters, if the probability of wrong fixing is not sufficiently small. The best integer-equivariant (BIE) estimator is optimal in the sense of minimizing the mean-squared error (MSE) of both the integer and real valued parameters, regardless of the precision of the float solution. However, like ILS, the BIE estimator comprises a search in the integer space of ambiguities, whose complexity grows exponentially with the number of ambiguities, which is not feasible for large-scale network solutions. To overcome this problem, a sequential BIE (SBIE) algorithm is proposed, which shows close to optimal performance while being part of the class with complexity of linear order. Numerical simulations are used to verify the performance of the SBIE algorithm.

1 INTRODUCTION

The precision of conventional GNSS systems can be considerably increased through the use of carrier-phase measurements, which can be tracked with millimeter accuracy, but are ambiguous. In the past two decades, research focussing on the estimation of these integer valued ambiguities was of great interest, reaching from the theoretical concept of estimation [1] over ambiguity validation [2] to highly reliable estimation schemes [3, 4]. As the reliability of integer ambiguity estimation strongly depends on the precise knowledge of error sources like atmospheric delays or instrumental hardware biases, one usually uses double difference measurements for differential positioning, which strongly suppresses most of the error sources. However, the availability of highly precise satellite clock and orbit corrections enables the use of an absolute positioning strategy with undifferenced measurements [5], referred to as precise point positioning. Obviously, in such an absolute positioning scheme, the above mentioned error sources are all present and have to be estimated in advance in order to adjust the GNSS measurements. For the joint estimation of satellite biases or orbit corrections, a large-scale network of reference stations with precisely known coordinates is required.

There exists a great variety of integer ambiguity estimators, reaching from a standard linear least-squares estimator (which is called the *float* solution) to non-linear estimation schemes like BS [6], integer aperture estimators [2, 7], and the ILS estimator, which minimizes the squared norm of the residuals of the linear GNSS model under the constraint of the ambiguity estimates being integer valued. A very efficient implementation of the latter scheme is given by the famous Least-squares AMBiguity Decorrelation Adjustment (LAMBDA) method [8]. Often, an integer valued ambiguity estimate resulting from BS or ILS is called *fixed* solution. A further, very promising estimator is the BIE estimator [9], which results from a joint minimization of the MSE of both integer ambiguities and real valued parameters over a more general class of estimators, which includes both float and fixed solution, and also the class of integer aperture estimators. However, just like ILS, the BIE estimator comprises a search in the integer space of ambiguities, which constitutes the major drawback of this estimator and disqualifies its use for a joint network parameter estimation with a large number of ambiguities and, therefore, for satellite bias estimation. In order to overcome this complexity problem, a suboptimal estimation scheme based on the BIE estimator can be used. In [10], a partial BIE estimation scheme was proposed, which combines BS, BIE and float solution, and uses each of them in the respective regime, where it performs close to optimum. In this paper, a *sequential* BIE estimation scheme is introduced, which shows very low complexity at the cost of only slightly decreased performance.

Outline: The remainder of this paper is organized as follows. In Section 2, the principle of BIE estimation is reviewed before introducing the SBIE algorithm. A more detailed description of the GNSS system model used for parameter estimation is given in Section 3, and finally some numerical results based on this system model are presented in Section 4.

2 GNSS PARAMETER ESTIMATION

In principle all GNSS problems (e.g., positioning or the estimation of code- and phase-biases) that include carrier-phase and, optionally, also code measurements or combinations thereof, can be cast as a system of linear equations in the form

$$\boldsymbol{\Psi} = \boldsymbol{A}\boldsymbol{N} + \boldsymbol{H}\boldsymbol{\xi} + \boldsymbol{\eta}_{\boldsymbol{\Psi}}, \tag{1}$$

where $\boldsymbol{\Psi} \in \mathbb{R}^q$ is the measurement vector, which contains undifferenced measurements, single- or double-difference combinations. The vectors $\boldsymbol{N} \in \mathbb{Z}^n$ and $\boldsymbol{\xi} \in \mathbb{R}^p$ denote the unknown integer ambiguities and real valued parameters, respectively, and the corresponding matrices $\boldsymbol{A} \in \mathbb{R}^{q\times n}$ and $\boldsymbol{H} \in \mathbb{R}^{q\times p}$ represent the linear system model. Finally, $\boldsymbol{\eta}_{\boldsymbol{\Psi}} \in \mathbb{R}^q$ is an additive Gaussian noise with zero mean and covariance matrix $\boldsymbol{Q}_{\boldsymbol{\eta}_{\boldsymbol{\Psi}}} \in \mathbb{R}^{q\times q}$.

Best Integer-Equivariant Estimation

As the name indicates, the class of integer-equivariant estimators is characterized by the integer remove-restore property, i.e., if the measurements are perturbed by an arbitrary number of cycles $\boldsymbol{A}\boldsymbol{z}, \forall \boldsymbol{z} \in \mathbb{Z}^n$, the solution for the ambiguity estimate is shifted by $\boldsymbol{z}$. Likewise a perturbation by $\boldsymbol{H}\boldsymbol{\zeta}, \forall \boldsymbol{\zeta} \in \mathbb{R}^p$, results in a shift of the real valued parameter vector by $\boldsymbol{\zeta}$, while the integer part is not affected. It can easily be shown, that all linear unbiased estimators fulfill this property [11], which makes the float estimator part of this class. Furthermore, admissible integer estimators [1] like BS or ILS are also part of this class by definition. Minimizing the MSE over the class of integer-equivariant estimators thus leads to an estimator (the BIE estimator) with at least equal performance compared to both the float and fixed solution [12].

From the definiton of the MSE criterion in [9] it follows, that the BIE estimator not only minimizes the mean-squared error in the Euclidean sense, but also in the metric of an *arbitrary* covariance matrix. Hence, the BIE estimates for ambiguities and real valued parameters are given by

$$\begin{bmatrix} \check{\boldsymbol{N}}_{\mathrm{BIE}} \\ \check{\boldsymbol{\xi}}_{\mathrm{BIE}} \end{bmatrix} = \arg\min_{\check{\boldsymbol{N}},\check{\boldsymbol{\xi}}} \mathrm{E}\left[\left\| \begin{bmatrix} \check{\boldsymbol{N}} \\ \check{\boldsymbol{\xi}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{\xi} \end{bmatrix} \right\|^2_{\boldsymbol{Q}^{-1}} \right], \quad \text{s.t.} \quad \boldsymbol{Q} \in \mathbb{R}^{(n+p)\times(n+p)}, \boldsymbol{Q} \succeq \boldsymbol{0}, \tag{2}$$

where $\succeq$ is defined is the sense of positive semi-defineteness, and the estimates $\check{\boldsymbol{N}}$ and $\check{\boldsymbol{\xi}}$ result from an integer-equivariant estimator.

Although the BIE estimator can be derived for an arbitrary noise contribution $\boldsymbol{\eta}_{\boldsymbol{\Psi}}$ in (1) [11], we restrict ourselves to Gaussian noise in this work. That being the case, the BIE estimator follows the three-step procedure, which is well known from integer (aperture) estimators [1, 2, 8]:

1. The integer property of the ambiguities $\boldsymbol{N}$ is discarded, i.e., a standard linear weighted least-squares adjustment is performed, which leads to the unbiased, Gaussian distributed float solutions $\hat{\boldsymbol{N}}$ and $\hat{\boldsymbol{\xi}}$ with the respective covariance matrices $\boldsymbol{Q}_{\hat{\boldsymbol{N}}} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{Q}_{\hat{\boldsymbol{\xi}}} \in \mathbb{R}^{p\times p}$, and cross-covariance matrix $\boldsymbol{Q}_{\hat{\boldsymbol{\xi}},\hat{\boldsymbol{N}}} \in \mathbb{R}^{p\times n}$.

2. A mapping is introduced, that allocates to each $\hat{\boldsymbol{N}}$ the BIE ambiguity solution $\check{\boldsymbol{N}}_{\mathrm{BIE}}$.

3. This ambiguity estimate is then used to adjust the float solution of the real valued parameters $\boldsymbol{\xi}$ in a least-squares sense, i.e.,

$$\check{\boldsymbol{\xi}}_{\mathrm{BIE}} = \hat{\boldsymbol{\xi}}_{\mathrm{BIE}} - \boldsymbol{Q}_{\hat{\boldsymbol{\xi}}\hat{\boldsymbol{N}}} \boldsymbol{Q}_{\hat{\boldsymbol{N}}}^{-1} \left(\hat{\boldsymbol{N}} - \check{\boldsymbol{N}}_{\mathrm{BIE}} \right). \quad (3)$$

The crucial point in the above procedure is the second step, which is given by

$$\check{\boldsymbol{N}}_{\mathrm{BIE}} = \sum_{\boldsymbol{z} \in \mathbb{Z}^n} \boldsymbol{z} w_{\boldsymbol{z}} \left(\hat{\boldsymbol{N}} \right) \text{ with } \sum_{\boldsymbol{z} \in \mathbb{Z}^n} w_{\boldsymbol{z}} \left(\hat{\boldsymbol{N}} \right) = 1, \quad (4)$$

i.e., the BIE ambiguity solution $\check{\boldsymbol{N}}_{\mathrm{BIE}}$ is a weighted sum of all vectors in the n-dimensional space of integers. The weighting coefficients in (4) are computed as

$$w_{\boldsymbol{z}} \left(\hat{\boldsymbol{N}} \right) = \frac{\exp \left(-\frac{1}{2} \left\| \hat{\boldsymbol{N}} - \boldsymbol{z} \right\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{N}}}^{-1}} \right)}{\sum_{\boldsymbol{z}' \in \mathbb{Z}^n} \exp \left(-\frac{1}{2} \left\| \hat{\boldsymbol{N}} - \boldsymbol{z}' \right\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{N}}^{-1}}} \right)}. \quad (5)$$

Because of the infinite sums in (4) and (5), the computation of the true BIE ambiguity estimates is not feasible. As a consequence, the sums are constrained to the set $\boldsymbol{\Theta}^d_{\hat{\boldsymbol{N}}}$, which contains all integer vectors within an ellipsoidal region around the float solution $\hat{\boldsymbol{N}}$ with radius d defined in the metric of the covariance matrix $\boldsymbol{Q}_{\hat{\boldsymbol{N}}}$. This choice preserves the integer-equivariance property of the estimator. The size of the search space is set according to a probabilistic criterion in [13], which is possible through exploiting the fact, that $\|\hat{\boldsymbol{N}} - \boldsymbol{N}\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{N}}}^{-1}}$ follows a central χ^2 distribution with n degrees of freedom. Obviously, if the probability of $\hat{\boldsymbol{N}}$ lying within the volume of $\boldsymbol{\Theta}^d_{\boldsymbol{N}}$ is fixed, the radius d and, therefore, the number of integer candidates gets smaller for increasing precision of the float solution.

In order to enable an efficient search of the integer candidates within $\boldsymbol{\Theta}^d_{\hat{\boldsymbol{N}}}$, some methods from the LAMBDA algorithm, namely the prior integer decorrelation and the triangularization of the covariance matrix $\boldsymbol{Q}_{\hat{\boldsymbol{N}}}$, which allows for a recursive tree-search formulation [8, 14], should be adapted. For a low complexity implementation, which it is aimed at in this contribution, the functional decorrelation in [15] shall be mentioned as an alternative to the iterative $\boldsymbol{Z}$ transformation of the LAMBDA method.

Sequential Best Integer-Equivariant Estimation

Although the integer decorrelation and tree-search reformulation dramatically reduce the complexity of finding the integer candidates within the search space $\boldsymbol{\Theta}^d_{\hat{\boldsymbol{N}}}$, the complexity of the search itself still grows exponentially with the number of ambiguities n. Therefore, a suboptimal, but from a computational point of view less demanding approach is now introduced, which combines the optimal BIE estimator with a sequential processing strategy. Instead of performing *one* n-dimensional search in the integer space of ambiguities, n *one-dimensional* searches are performed, i.e., a separate BIE estimation is done for each ambiguity. Note, that only the second step of the three-step framework is affected.

Like in the BS algorithm, the ambiguities are sequentially estimated, starting with the first ambiguity and conditioning each ambiguity on the ones, that have already been estimated. The jth conditioned ambiguity is given by

$$\hat{N}_{j|\mathcal{J}} = \hat{N}_j - \sum_{l=1}^{j-1} \sigma_{\hat{N}_j \hat{N}_{l|\mathcal{L}}} \sigma^{-2}_{\hat{N}_l} \left(\hat{N}_{l|\mathcal{L}} - \check{N}_{l,\mathrm{SBIE}} \right), \quad \forall j \in \{1, \ldots, n\}, \quad (6)$$

where $\sigma_{\hat{N}_j \hat{N}_{l|\mathcal{L}}}$ denotes the covariance between $\hat{N}_j$ and $\hat{N}_{l|\mathcal{L}}$, $\sigma^2_{\hat{N}_l}$ is the variance of $\hat{N}_l$, and the set $\mathcal{J} = \{1, \ldots, j-1\}$ for $j > 0$ and $\mathcal{J} = \varnothing$ for $j = 0$ ($\mathcal{L}$ is defined accordingly). The variances and covariances are accessible from the $\boldsymbol{LDL}^{\mathrm{T}}$ decomposition of $\boldsymbol{Q}_{\hat{\boldsymbol{N}}}$ [6], which is already available from the integer decorrelation.

The final SBIE ambiguity estimates, also required for (6) itself, are computed from (4) and (5), adjusted to the one-dimensional case, from the conditional float ambiguity estimates $\hat{N}_{l|\mathcal{L}}$ with variance $\sigma^2_{\hat{N}_{l|\mathcal{L}}}$ for all $l \in \{1, \ldots, n\}$, as

$$\check{N}_{l,\mathrm{SBIE}} = \sum_{z \in \boldsymbol{\Theta}^d_{\hat{N}_{l|\mathcal{L}}}} z \, w_z \left(\hat{N}_{l|\mathcal{L}} \right). \quad (7)$$

In contrast to the multivariate case, the search for the integer candidates itself is a trivial task in one-dimensional space, as one simply has to find all scalar integers within a given interval

$$\boldsymbol{\Theta}^d_{\hat{N}_{l|\mathcal{L}}} = z \in \left[\hat{N}_{l|\mathcal{L}} - d\sigma_{\hat{N}_{l|\mathcal{L}}}, \hat{N}_{l|\mathcal{L}} + d\sigma_{\hat{N}_{l|\mathcal{L}}} \right], \quad \text{s.t.} \quad z \in \mathbb{Z}. \quad (8)$$

The normalized size of the search interval d does not depend on the variance of the conditioned float solution and – as already mentioned – can be derived from the probabilistic criterion

$$\Pr \left(\frac{\left| N_l - \hat{N}_{l|\mathcal{L}} \right|^2}{\sigma^2_{\hat{N}_{l|\mathcal{L}}}} \leq d^2 \right) = 1 - \varepsilon \quad (9)$$

for a fixed value of ε, as $\left| N_l - \hat{N}_{l|\mathcal{L}} \right|^2 / \sigma^2_{\hat{N}_{l|\mathcal{L}}}$ follows a central χ^2 distribution with one degree of freedom, if the randomness of $\check{N}_{j,\mathrm{SBIE}}$ is neglected $\forall j \in \mathcal{L}$.

After the sequential estimation, the ambiguities are stacked into a vector

$$\check{\boldsymbol{N}}_{\mathrm{SBIE}} = \left[\check{N}_{1,\mathrm{SBIE}}, \ldots, \check{N}_{n,\mathrm{SBIE}} \right]^{\mathrm{T}}, \quad (10)$$

which is then used to adjust the float solution of the real valued parameters $\hat{\boldsymbol{\xi}}$ according to the third step of the three-step framework.

Obviously, this SBIE estimation allows for a complexity reduction from *exponential* order for the true BIE estimator to *linear* order. Therefore, the complexity of SBIE is comparable to the one of the BS algorithm, and, coming along with that, also much lower than the one of ILS. The loss of performance resulting from the suboptimality is to be numerically evaluated in Section 4.

Note, that the resulting SBIE ambiguity estimates are not unique, but depend on the ordering of the (decorrelated) ambiguities. In the simulations below, the order is chosen such, that the precision of the float ambiguity estimates "decreases" with increasing index, i.e., in each step the ambiguity with the float solution showing the smallest conditional variance is chosen next.

Optimality of SBIE

Despite its general suboptimality, there exist three cases, in which the SBIE estimator is equal to or converges to the MSE optimal BIE solution.

1. *The covariance matrix $\boldsymbol{Q}_{\hat{\boldsymbol{N}}}$ is diagonal:* In that case the SBIE algorithm can be shown to be equivalent to the true BIE algorithm. As the single ambiguities are uncorrelated, the sequential estimation of the ambiguities gets a *separate* estimation, which should be intuitive to be optimal. This property represents a further motivation for a prior integer decorrelation, which diagonalizes $\boldsymbol{Q}_{\hat{\boldsymbol{N}}}$ as far as possible.

2. *The float solution $\hat{\boldsymbol{N}}$ shows very low precision:* The integer grid is very dense compared to the probability density of the float solution $\hat{\boldsymbol{N}}$. Therefore, in the limit case, for each ambiguity there exists an integer larger than the float solution corresponding to each integer smaller than the float solution with the same distance to $\hat{N}_{l|\mathcal{L}}$, which average to the (conditioned) float solution itself (see Figure 1), which is MSE optimal in that regime [11].

3. *The float solution $\hat{\boldsymbol{N}}$ shows very high precision:* The probability density of the float solution $\hat{\boldsymbol{N}}$ is very peaked compared to the integer grid, which causes a SBIE solution, that automatically converges to the BS solution, as the weighting coefficients in (7) approach binary values (see Figure 1). For vanishing probability of wrong fixing, i.e., in the high precision regime, the BS solution is also MSE optimal.

The optimality of SBIE for these three cases is proven analytically in the Appendix.

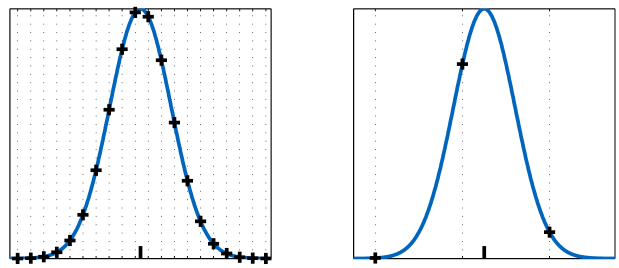

Fig. 1 The weights of the integer candidates (dotted lines), which are directly proportional to the value of the Gauss–curve centered around the float solution, are illustrated for low (left subfigure) and high (right subfigure) precision. We can see, that for low precision there is always a pair of candidates with approximately the same weights averaging to the float solution itself, while for high precision the weights approach binary values.

3 SIMULATION SETUP

GNSS System Model

Assuming, that the hardware biases of the GNSS measurements are not link dependent, but can rather be modeled as a sum of satellite and receiver specific biases, the absolute code- $\rho^k_{r,m}(t)$ and carrier-phase $\lambda_m \varphi^k_{r,m}(t)$ measurements at time t for satellite k and user r on frequency m with wavelength λ_m can be modeled as [3]

$$\begin{aligned} \rho^k_{r,m}(t) &= g^k_{r,m}(t) + q^2_{1m} I^k_r(t) + b_{r,m} + b^k_m + \eta^k_{r,m}(t) \\ \lambda_m \varphi^k_{r,m}(t) &= g^k_{r,m}(t) - q^2_{1m} I^k_r(t) + \beta_{r,m} + \beta^k_m \\ &\quad + \lambda_m N^k_{r,m} + \epsilon^k_{r,m}(t). \end{aligned} \tag{11}$$

The ionospheric slant delay (phase advance) on the first frequency is denoted by $I^k_r(t)$ and used with scaling factor $q_{1m} = f_1/f_m$ for frequency m. The constant code- and phase-biases on each frequency are given by $b_{r,m}$, b^k_m, $\beta_{r,m}$ and β^k_m for receivers and satellites, respectively, and the integer ambiguities for each link are $N^k_{r,m}$. The additive noise contributions $\eta^k_{r,m}(t)$ and $\epsilon^k_{r,m}(t)$ for code- and carrier-phase measurements are modeled as zero mean white Gaussian processes, i.e., a possible multipath effect is already corrrected for [16].

The geometry terms $g^k_{r,m}(t)$ can further be rewritten as

$$\begin{aligned} g^k_{r,m}(t) &= \boldsymbol{e}^{k,\mathrm{T}}_r(t)\left(\boldsymbol{x}_r(t) - \boldsymbol{x}^k_r(t)\right) + c\left(\delta\tau_r(t) - \delta\tau^k_r(t)\right) \\ &\quad + M\left(E^k_r(t)\right) T_{\mathrm{z},r}(t), \end{aligned} \tag{12}$$

with $\boldsymbol{e}^k_r(t) \in \mathbb{R}^3$ being a unit vector pointing from satellite k to receiver r, $\boldsymbol{x}_r(t) \in \mathbb{R}^3$ the receiver position at the time of reception t, and $\boldsymbol{x}^k_r(t) \in \mathbb{R}^3$ the satellite position at the *time of transmission* to receiver r. Furthermore, the clock offsets of receiver and satellite are denoted by

$\delta\tau_r(t)$ and $\delta\tau_r^k(t)$ at the time of reception and transmission, respectively, and the tropospheric delay is modeled as the product of the zenith delay $T_{z,r}(t)$ at receiver r and the mapping function $M\left(E_r^k(t)\right)$ depending on the elevation angle $E_r^k(t)$ between satellite k and user r.

Further effects like phase-wind-up and phase-center variations are assumed to be completely compensated for in advance.

Parameter Mappings

A joint estimation of all hardware biases, ambiguities, clock offsets and atmospheric errors is not feasible due to the rank deficiency of the system of equations (11). Therefore, the three-step parameter mapping strategy from [17] is applied before the parameter estimation. Firstly, the code-biases $b_{r,m}$ and b_m^k are equivalently rewritten as a frequency dependent and a frequency independent component for each receiver and satellite, where the former ones are treated as virtual additional ionospheric delays and the latter ones as additional clock offsets.

In the second step, one of the satellites is chosen as a reference satellite. The phase bias of that satellite is absorbed in the receiver phase-biases $\beta_{r,m}$, and all satellite phase-biases β_m^k are only estimated as the offset to the bias of that reference satellite. The same strategy is applied to the receiver and satellite clock errors $\delta\tau_r(t)$ and $\delta\tau_r^k(t)$.

As a last step, the linear dependencies between the remaining phase-biases and the integer ambiguities $N_{r,m}^k$ are removed via a Gaussian elimination. Thereby, one set of ambiguities is mapped to the phase-biases and becomes real valued, while a second set of ambiguities is exclusively mapped to other ambiguities, which preserves the integer property.

After all parameter mappings the system model (11) can be written in the form of (1), and the presented three-step framework can be applied, either with BS, ILS, BIE or SBIE.

Estimation Strategy

Besides the measurement model (11), the temporal evolution of the parameters in the state vector can be modeled as a Gauss-Markov process with white Gaussian process noise. Apparently, for all constant parameters, like biases and integer ambiguities, no noise is present. After a least-squares initialization the MSE-optimal estimates can be computed recursively with a Kalman filter, which minimizes the MSE for the class of linear estimators. Based on the Gaussian distributed estimates resulting from the update-step of the Kalman filter, further, non-linear estimators like SBIE can be applied, i.e., the outcome of the Kalman filter serves as the float solution of the three-step framework of Section 2. As the BIE (and also SBIE) estimates do not follow a Gaussian distribution, a feedback-loop to the Kalman filter cannot be set up to fasten the convergence.

The great advantage of using a Kalman filter for evaluating the performance of the SBIE estimator compared to other schemes is, that it offers a wide range of precision characteristics of the float solution, starting with very low precision and ending up at very high precision after sufficient convergence. Thus, evaluating the performance of the different estimators as a function of the epochs of the Kalman filter also allows for an interpretation of the performance depending on the precision of the float solution.

4 NUMERICAL RESULTS

For all simulations, pseudorange and carrier-phase measurements on the two frequencies $f_1 = 154 \cdot 10.23\,\mathrm{MHz}$ and $f_2 = 120 \cdot 10.23\,\mathrm{MHz}$ are modeled, where the measurement noise is chosen as $\sigma_\rho = 1\,\mathrm{m}$ and $\sigma_\varphi = 2\,\mathrm{cm}$, respectively. The time interval between two subsequent epochs is chosen as $\triangle t = 1\,\mathrm{s}$.

Small Network MSE Evaluation

A small toy-example with only $R = 2$ reference stations, each having a line of sight connection to $K = 6$ satellites is considered. After the parameter mappings this results in $n = 10$ ambiguities and $p = 35$ real valued parameters, which also allows for applying the two search-based estimators, namely BIE and ILS, due to the small number of ambiguities. The performance of the different estimation schemes is compared via the weighted MSE

$$\begin{aligned}\mathrm{MSE}_{\{\mathrm{F,BS,ILS,SBIE,BIE}\}}(t) = \\ \mathrm{E}\left[\left\|\begin{bmatrix}\tilde{\boldsymbol{N}}(t)\\ \tilde{\boldsymbol{\xi}}(t)\end{bmatrix} - \begin{bmatrix}\boldsymbol{N}\\ \boldsymbol{\xi}(t)\end{bmatrix}\right\|^2_{\boldsymbol{Q}_\mathrm{F}^{-1}(t)}\right] \\ \text{with} \quad \tilde{\boldsymbol{N}}(t) \in \left\{\hat{\boldsymbol{N}}(t), \check{\boldsymbol{N}}_{\{\mathrm{BS,ILS,SBIE,BIE}\}}(t)\right\}, \\ \text{and} \quad \tilde{\boldsymbol{\xi}}(t) \in \left\{\hat{\boldsymbol{\xi}}(t), \check{\boldsymbol{\xi}}_{\{\mathrm{BS,ILS,SBIE,BIE}\}}(t)\right\}, \end{aligned} \quad (13)$$

where $\boldsymbol{Q}_\mathrm{F}(t) \in \mathbb{R}^{(n+p)\times(n+p)}$ is the covariance matrix of the float solution at time t. By definition, the MSE of the float solution is constant with the number of unknowns

$$\mathrm{MSE}_\mathrm{F}(t) = n + p. \quad (14)$$

The MSE curves resulting from Monte-Carlo simulations with 10^4 realizations are plotted against epochs in Figure 2. Due to the high probability of wrong fixing, the performance of ILS and BS is worse compared to the float solution in the first epochs, but will converge to the optimal BIE solution with vanishing probability of wrong fixing (not depicted here). The increase of the MSE in the first epochs can be ascribed to the weighting with the inverse covariance matrix with decreasing variances. By definition, the BIE solution is optimal, though it appears to be worse than the float solution at very low precision, which is caused by the sloppy approximation of the ambiguity estimate

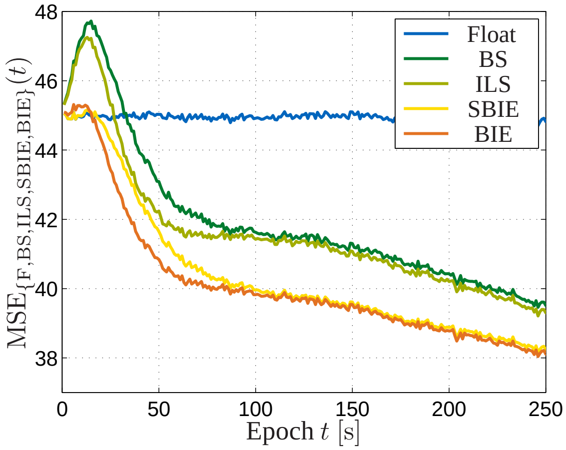

Fig. 2 MSE curves of float solution, BS, ILS, SBIE and BIE estimates plotted agains epochs t.

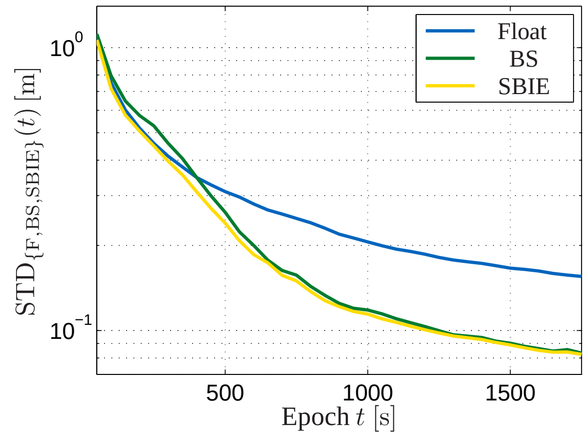

Fig. 3 Mean standard deviations of the satellite phase-bias estimates for float solution, BS, and SBIE plotted agains epochs t.

$\check{\boldsymbol{N}}_{\mathrm{BIE}}$ in that regime (not critical for higher precision, where a smaller number of candidates is required). As already stated, the SBIE estimator shows (quasi-) optimal performance for low and high precision. The only penalty appears – as expected – in between these two cases, yet it is small enough, that SBIE is still superior to ILS. Note, that approximating the SBIE ambiguity estimates in the first epochs is much easier and, thus, can be done more precisely than the approximation of the BIE ambiguity estimates.

Satellite Phase-Bias Estimation

A global network of $R = 20$ reference stations is considered for satellite bias estimation, and the satellite constellation is assumed to follow a Walker 27/3/1 constellation, i.e., the Galileo setup. The elevation mask is set to $5\,\mathrm{deg}$, and Niell's model [18] is used for modeling tropospheric slant delays. Compared to the previous toy-example, satellite orbit errors and error rates are included in the parameter vector, which are described in the RIC coordinate frame (radial, in-track, and cross-track) [16], and follow Newton's laws of motion. For the simulated time span of $30\,\mathrm{min}$, a total number of ambiguities $n = 228$ is reached, considering only those satellites at each receiver, which are visible over the whole period. Therefore, ILS and BIE are excluded in the analysis, as their respective searches are too complex.

As the float solution as well as BS and SBIE are unbiased estimators, their performance can be compared using the mean (with respect to different satellites and frequencies) standard deviation of the satellite phase-bias estimates

$$\mathrm{STD}^2_{\{\mathrm{F,BS,SBIE}\}}(t) = \mathrm{E}_{k,m}\left[\left(\tilde{\beta}^k_m(t) - \beta^k_m\right)^2\right]$$

$$\text{with} \quad \tilde{\beta}^k_m(t) \in \left\{\hat{\beta}^k_m(t), \check{\beta}^k_{m,\{\mathrm{BS,SBIE}\}}(t)\right\}. \qquad (15)$$

One obvious result from Figure 3 is, that by somehow taking into consideration the integer nature of the ambiguities, the accuracy of the float bias estimates can be considerably improved, if the precision of the float solution itself is not too low. A great advantage of SBIE (and also BIE) is, that one does not have to think about which ambiguities to fix or when to fix a certain ambiguity, and setting up some decision criterion, as SBIE automatically converges to the respective best solution.

5 CONCLUSION

A novel GNSS parameter estimation scheme was proposed, which combines the MSE-optimal BIE estimator with the sequential estimation strategy known from BS. Thereby, the complexity was reduced from exponential to linear order, as the computationally demanding search operation was replaced by trivial, one-dimensional searches. In numerical simulations the close to optimum performance of SBIE was proven true. These properties not only provide for an applicability in large-scale networks, but also for applications, which require low complexity, such as in smartphones, automobile or maritime devices for carrier-phase based positioning.

ACKNOWLEDGMENT

The authors would like to thank the German Federal Ministry of Economic Affairs and Technology (BMWi) and the German Aerospace Center (DLR) for a financial grant (FKZ: 50NA1104).

APPENDIX

Proof: SBIE is optimal for $\boldsymbol{Q}_{\hat{\boldsymbol{N}}}$ diagonal: the exponential function in the computation of the BIE ambiguity estimate

(5) can be split to

$$\exp\left(-\frac{1}{2}\left\|\hat{\boldsymbol{N}}-\boldsymbol{z}\right\|^2_{\boldsymbol{Q}^{-1}_{\hat{\boldsymbol{N}}}}\right) = \prod_{l=1}^{n}\exp\left(-\frac{1}{2\sigma^2_{\hat{N}_l}}\left(\hat{N}_l - z_l\right)^2\right) = \prod_{l=1}^{n} e_l\left(z_l\right), \qquad (16)$$

which is a product of factors $e_l\left(z_l\right)$ only depending on the lth ambiguity. Hence, the BIE estimate for the jth ambiguity according to (4), (5) and (16) is given by

$$\begin{aligned}\check{N}_{j,\mathrm{BIE}} &= \sum_{\boldsymbol{z}\in\mathbb{Z}^n} z_j \frac{\prod_{l=1}^{n} e_l\left(z_l\right)}{\sum_{\boldsymbol{z}'\in\mathbb{Z}^n}\prod_{l=1}^{n} e_l\left(z'_l\right)} \\ &= \frac{1}{\sum_{\boldsymbol{z}'\in\mathbb{Z}^n}\prod_{l=1}^{n} e_l\left(z'_l\right)} \cdot \sum_{\boldsymbol{z}\setminus z_j\in\mathbb{Z}^{n-1}} \prod_{\substack{l=1\\ l\neq j}}^{n} e_l\left(z_l\right) \sum_{z_j\in\mathbb{Z}} z_j e_j\left(z_j\right) \\ &= \sum_{z_j\in\mathbb{Z}} z_j \frac{e_j\left(z_j\right)}{\sum_{z'_j\in\mathbb{Z}} e_j\left(z'_j\right)}, \end{aligned} \qquad (17)$$

which corresponds to a separate estimation of all ambiguities with a one-dimensional BIE estimator, which is again equal to the SBIE estimator for uncorrelated float ambiguity estimates.

Proof: SBIE is optimal for very low precision of $\hat{\boldsymbol{N}}$: as the integer grid gets very dense compared to the probability density of the float solution, the summation in (7) can be replaced by proper integration in the limit case (an infinite search space is assumed)

$$\check{N}_{l,\mathrm{SBIE}} = \int_{\mathbb{R}} z\, w_z\left(\hat{N}_{l|\mathcal{L}}\right) \mathrm{d}z = \hat{N}_{l|\mathcal{L}}, \qquad (18)$$

which is equal to the conditioned float solution $\hat{N}_{l|\mathcal{L}}$, as $\int_{\mathbb{R}} w_z\left(\hat{N}_{l|\mathcal{L}}\right)\mathrm{d}z = 1$ by definition and $w_z\left(\hat{N}_{l|\mathcal{L}}\right)$ is symmetric with respect to $\hat{N}_{l|\mathcal{L}}$. Together with (6) follows

$$\check{N}_{l,\mathrm{SBIE}} = \hat{N}_l, \quad \forall l \in \{1,\ldots,n\}, \qquad (19)$$

which concludes the proof, as the float solution is optimal.

Proof: SBIE is optimal for very high precision of $\hat{\boldsymbol{N}}$: according to (5), the weighting coefficients $w_z\left(\hat{N}_{l|\mathcal{L}}\right)$ for (7) are given by (again assuming an infinite search space)

$$\begin{aligned} w_z\left(\hat{N}_{l|\mathcal{L}}\right) &= \frac{\exp\left(-\frac{1}{2\sigma^2_{\hat{N}_{l|\mathcal{L}}}}|z-\hat{N}_{l|\mathcal{L}}|^2\right)}{\sum_{z'\in\mathbb{Z}}\exp\left(-\frac{1}{2\sigma^2_{\hat{N}_{l|\mathcal{L}}}}|z'-\hat{N}_{l|\mathcal{L}}|^2\right)} \\ &= \frac{1}{1+\sum_{\substack{z'\in\mathbb{Z}\\ z'\neq z}}\exp\left(-\frac{1}{2\sigma^2_{\hat{N}_{l|\mathcal{L}}}}\left(|z'-\hat{N}_{l|\mathcal{L}}|^2-|z-\hat{N}_{l|\mathcal{L}}|^2\right)\right)}. \end{aligned} \qquad (20)$$

Let the rounding operator be denoted by $[\cdot]$. From (20) it follows, that

$$\lim_{\sigma_{\hat{N}_{l|\mathcal{L}}}\to 0} w_{\left[\hat{N}_{l|\mathcal{L}}\right]}\left(\hat{N}_{l|\mathcal{L}}\right) = 1, \qquad (21)$$

as $\left|\left[\hat{N}_{l|\mathcal{L}}\right]-\hat{N}_{l|\mathcal{L}}\right|^2 \leq |z'-\hat{N}_{l|\mathcal{L}}|^2, \forall z'\in\mathbb{Z}\setminus\left[\hat{N}_{l|\mathcal{L}}\right]$. As a consequence of $\sum_{z\in\mathbb{Z}} w_z\left(\hat{N}_{l|\mathcal{L}}\right) = 1$,

$$\lim_{\sigma_{\hat{N}_{l|\mathcal{L}}}\to 0} w_z\left(\hat{N}_{l|\mathcal{L}}\right) = 0, \quad \forall z\in\mathbb{Z}\setminus\left[\hat{N}_{l|\mathcal{L}}\right]. \qquad (22)$$

Inserting (21) and (22) in (7) proves the equivalence of SBIE and conditional rounding (i.e., BS) for the high precision regime, which is optimal there.

REFERENCES

[1] P. J. G. Teunissen, “Towards a unified theory of GNSS ambiguity resolution,” *Journal of Global Positioning Systems*, vol. 2, no. 1, pp. 1–12, December 2003.

[2] S. Verhagen, “The GNSS integer ambiguities: estimation and validation,” Dissertation, Technische Universiteit Delft, June 2004.

[3] P. Henkel, “Reliable carrier phase positioning,” Dissertation, Technische Universität München, October 2010.

[4] P. Henkel and C. Günther, “Reliable Integer Ambiguity Resolution with Multi-Frequency Code Carrier Linear Combinations,” *Journal of Global Positioning Systems*, vol. 9, no. 2, pp. 90–103, 2010.

[5] J. F. Zumberge, M. B. Heflin, D. C. Jefferson, M. M. Watkins, and F. H. Webb, “Precise point positioning for the efficient and robust analysis of GPS data from large networks,” *J. Geophys. Res.*, vol. 102, pp. 5005–5018, 1997.

[6] P. J. G. Teunissen, “GNSS ambiguity bootstrapping: Theory and application,” in *Proceedings of Int. Symp. on Kinematic Systems in Geodesy, Geomatics and Navigation*, Banff, Canada, August–September 2001, pp. 246–254.

[7] ——, “Integer aperture GNSS ambiguity resolution,” *Artificial Satellites*, vol. 38, no. 3, pp. 79–88, 2003.

[8] ——, “The least-squares ambiguity decorrelation adjustment: a method for fast GPS integer ambiguity estimation,” *Journal of Geodesy*, vol. 70, pp. 65–82, 1995.

[9] ——, “GNSS Best Integer Equivariant Estimation,” in *A Window on the Future of Geodesy*, ser. International Association of Geodesy Symposia, 2005, vol. 128, pp. 422–427.

[10] Z. Wen, P. Henkel, A. Brack, and C. Günther, "Satellite Bias Determination with Global Station Network and Best Integer-Equivariant Estimation," in *Proceedings of the ION GNSS International Technical Meeting 2012*, Nashville, Tennessee, US, 2012.

[11] P. J. G. Teunissen, "Theory of integer equivariant estimation with application to GNSS," *Journal of Geodesy*, vol. 77, pp. 402–410, 2003.

[12] S. Verhagen and P. J. G. Teunissen, "Performance comparison of the BIE estimator with the float and fixed GNSS ambiguity estimators," in *A Window on the Future of Geodesy*, ser. International Association of Geodesy Symposia, 2005, vol. 128, pp. 428–433.

[13] P. J. G. Teunissen, "On the Computation of the Best Integer Equivariant Estimator," *Artificial Satellites*, vol. 40, no. 3, pp. 161–171, 2005.

[14] P. D. Jonge and C. Tiberius, "The LAMBDA method for integer ambiguity estimation: implementation aspects," *Delft Geodetic Computing Centre LGR Series*, vol. 12, 1996.

[15] C. Günther and P. Henkel, "Integer Ambiguity Estimation for Satellite Navigation," *IEEE Transactions on Signal Processing*, vol. 60, no. 7, pp. 3387–3393, July 2012.

[16] Z. Wen, P. Henkel, M. Davaine, and C. Günther, "Satellite Phase and Code Bias Estimation with Cascaded Kalman Filter," in *Proceedings of European Navigation Conference (ENC)*, London, U.K., 2011.

[17] P. Henkel, Z. Wen, and C. Günther, "Estimation of satellite and receiver biases on multiple Galileo frequencies with a Kalman filter," in *Proceedings of the ION ITM 2010*, 2010, pp. 1067–1074.

[18] A. E. Niell, "Global mapping functions for the atmosphere delay at radio wavelengths," *Journal of Geophysical Research*, vol. 101, no. B2, pp. 3227–3246, Feb. 1996.